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Statistical hydromechanics of disperse systems. Part 2. Solution of the kinetic equation for suspended particles

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To solve the kinetic equation for particles of a monodisperse two-phase mixture the method of successive approximations is developed; this resembles in its main features the well-known Chapman–Enskog method in the kinetic theory of gases. This method is applicable for a mixture whose state differs slightly from the equilibrium, i.e., when time and space derivatives of the dynamic variables describing the mean flow of both phases of the mixture are sufficiently small. Accordingly, the solution obtained is valid when the time and space scales of the mean flow exceed considerably those for random pseudo-turbulent motion of particles and a fluid. The conservation equations for determination of all the dynamic variables are formulated in approximations which have the same meaning as those of Euler and Navier & Stokes in hydromechanics of one-phase media.

1. Introduction

The equations of mass and momentum conservation in the mean flow of a monodisperse system, whose phases are regarded as interpenetrating interacting continua, were derived in the first part of this paper (Buyevich 1971), henceforth denoted by I. These equations include various terms depending upon properties of random motion of particles and a fluid which must be expressed through the dynamic variables describing the mean flow. The later problem was considered in I only for a disperse system in the equilibrium state when the dynamic variables do not depend upon time and co-ordinates. In the general case, representations of those terms are unknown and the problem of their determination arises, for which purpose the kinetic equation for suspended particles has to be solved. Obviously, this problem is similar to that encountered in the kinetic theory of gases while formulating hydromechanic equations for a gas as for a continuum.

It is this problem that is treated below. We consider here monodisperse 'collisionless' systems where direct interparticle collisions are absent or, at any rate, play only a minor role in the momentum and energy exchange between particles. We employ also all the assumptions and notation of the previous work I without further comment. Moreover, we assume here for simplicity that quantities defining random motion inside a disperse system in the equilibrium state are represented already as functions of the dynamic variables. This can be done in principle by means of the technique proposed in I.

It is convenient to rewrite the conservation equations for both phases of a disperse system in the following way:

$$\left.\begin{aligned}
&\frac{D\langle\rho\rangle}{Dt} = -\langle\rho\rangle\frac{\partial \mathbf{w}}{\partial \mathbf{r}}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \mathbf{w}\rangle\frac{\partial}{\partial \mathbf{r}},\\
&\frac{D\langle \mathbf{w}\rangle}{Dt} = -\frac{1}{\langle\rho\rangle}\frac{\partial(\langle\rho\rangle\boldsymbol{\theta})}{\partial \mathbf{r}} + \langle \mathbf{H}\rangle, \quad \boldsymbol{\theta} = \langle \mathbf{w}' * \mathbf{w}'\rangle,\\
&\frac{D\langle\rho\rangle}{Dt} = -\langle \mathbf{u}\rangle\frac{\partial\langle\rho\rangle}{\partial \mathbf{r}} + (1-\langle\rho\rangle)\frac{\partial\langle \mathbf{v}\rangle}{\partial \mathbf{r}} + \frac{\partial \mathbf{q}}{\partial \mathbf{r}}, \quad \mathbf{q} = -\langle\rho' \mathbf{v}'\rangle, \quad \mathbf{u} = \mathbf{v} - \mathbf{w},\\
&\frac{D\langle \mathbf{v}\rangle}{Dt} = -\left(\langle \mathbf{u}\rangle\frac{\partial}{\partial \mathbf{r}}\right)\langle \mathbf{v}\rangle - \frac{1}{d_0(1-\langle\rho\rangle)}\frac{\partial\langle p\rangle}{\partial \mathbf{r}} - \mathbf{T} + \frac{2\nu_0}{1-\langle\rho\rangle}\frac{\partial \mathbf{E}}{\partial \mathbf{r}} + \mathbf{h},\\
&\mathbf{T} = \frac{1}{1-\langle\rho\rangle}\left\{\frac{\partial \mathbf{q}}{\partial t} + \left(\mathbf{q}\frac{\partial}{\partial \mathbf{r}}\right)\langle \mathbf{v}\rangle + \frac{\partial}{\partial \mathbf{r}}(\langle \mathbf{v}\rangle * \mathbf{q}) + \frac{\partial}{\partial \mathbf{r}}(1-\langle\rho\rangle)\langle \mathbf{v}' * \mathbf{v}'\rangle\right\},\\
&\mathbf{E} = \left(S + \frac{1}{2}\frac{d^2S}{d\langle\rho\rangle^2}\langle\rho'^2\rangle\right)\langle \mathbf{e}\rangle + \frac{dS}{d\langle\rho\rangle}\langle\rho' \mathbf{e}'\rangle, \quad S = S(\langle\rho\rangle),\\
&\mathbf{h} = \mathbf{g} - \frac{1}{\chi}\frac{\langle\rho\rangle}{(1-\langle\rho\rangle)}(\langle \mathbf{H}\rangle - \mathbf{g}), \quad \chi = \frac{d_0}{d_1}.
\end{aligned}\right\} \qquad (1.1)$$

Various averaged quantities of the type of $\langle\phi', \psi'\rangle$, where ϕ', ψ' are any pseudo-turbulent pulsations, appearing in (1.1) can be expressed in terms of $\boldsymbol{\theta}$ by means of the relations

$$\left.\begin{aligned}
&\langle\phi'\psi'\rangle = R[\phi, \psi]\,\theta, \quad \langle\phi'\psi_j'\rangle = R_i[\phi, \boldsymbol{\psi}]\,\theta_{ij},\\
&\langle\phi_i'\psi_j'\rangle = R_{ik}[\phi, \boldsymbol{\psi}]\,\theta_{kj}, \quad \theta = \operatorname{tr}\boldsymbol{\theta} = \boldsymbol{\theta}_{ii},
\end{aligned}\right\} \qquad (1.2)$$

which play the role of the 'constitutive' equations for the disperse system under consideration.

The kinetic equation for suspended particles has the form

$$\frac{Df}{Dt} + \mathbf{w}'\frac{\partial f}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{w}'}\left[\left(\mathbf{H}^* - \frac{D\langle \mathbf{w}\rangle}{Dt}\right)f\right] - \left(\frac{\partial f}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \left(\frac{\partial}{\partial \mathbf{r}} * \langle \mathbf{w}\rangle\right)$$
$$+ \left(\frac{\partial}{\partial \mathbf{w}'} * \frac{\partial}{\partial \mathbf{w}'}\right) : (\mathbf{A}f) = 0, \quad \mathbf{H}^* = \langle \mathbf{H}\rangle - \mathbf{c}\mathbf{w}'. \qquad (1.3)$$

By definition, we also have relationships between the distribution function f, normalized to the number concentration of particles n, and the quantities $\langle\rho\rangle$, $\langle \mathbf{w}\rangle, \boldsymbol{\theta}$:

$$\langle\rho\rangle = \sigma n = \sigma\int f d\mathbf{w}', \quad \langle \mathbf{w}\rangle = \frac{1}{n}\int \mathbf{w} f d\mathbf{w}', \quad \boldsymbol{\theta} = \frac{1}{n}\int(\mathbf{w}' * \mathbf{w}')f d\mathbf{w}'. \qquad (1.4a, b, c)$$

It is obvious from the physical point of view that the function f must meet the requirement

$$\lim_{|\mathbf{w}'|\to\infty} |\mathbf{w}'|^{\alpha} f = 0 \quad (\alpha > 0). \qquad (1.5)$$

The explicit representations for the mean force $\langle \mathbf{H}\rangle$ acting on the unit mass of a particle and for the tensor $\mathbf{c}$ involved in (1.3) are not essential for our treatment. Some possible approximate expressions for them are listed in I (see also equations (3.6) below). All the quantities R appearing in (1.2) are defined by properties of equilibrium pseudo-turbulence and can be readily found as the solutions of (1.2) relating to the equilibrium state, as has been pointed out in I. Therefore, in accordance with previous assumptions, they can be also regarded as known functions of the dynamic variables.

To make (1.1) completely definite in the general case, it is necessary to solve (1.3) for a non-equilibrium state of a disperse system, to calculate the tensor $\boldsymbol{\theta}$ from (1.4*c*) and to express pseudo-turbulent quantities involved in (1.1) in terms of the dynamic variables in accordance with (1.2). These are the main aims which are pursued below.

2. The 'equilibrium' distribution function

Let us begin with a calculation of the distribution function in the equilibrium state and with a determination of the tensor **A** describing the diffusion in the velocity space. For the state mentioned, equations (1.1) take the form

$$\langle \mathbf{H} \rangle^0 = 0, \quad -\nabla \langle p \rangle + d_0(1-\langle \rho \rangle)\,\mathbf{h}^0 = 0, \tag{2.1}$$

where the superscript zero indicates that the corresponding quantity is related to the equilibrium state when the dynamic variables are constant (except for the mean pressure, of course, which may depend linearly on co-ordinates). Similarly, the kinetic equation (1.3) is transformed into

$$-\frac{\partial}{\partial \mathbf{w}'}(\mathbf{c}^0\mathbf{w}'f^0) + \left(\frac{\partial}{\partial \mathbf{w}'} * \frac{\partial}{\partial \mathbf{w}'}\right) : (\mathbf{A}f^0) = 0. \tag{2.2}$$

Expressions for $\langle \mathbf{H} \rangle^0$ and $\mathbf{c}^0$ can be easily obtained from those for $\langle \mathbf{H} \rangle$ and $\mathbf{c}$, quantities of the type of $\langle \phi' \psi' \rangle$ being substituted for by their known equilibrium values $\langle \phi' \psi' \rangle^0$, and terms proportional to derivatives of the dynamic variables being dropped altogether (see for example, equations (3.6) below).

It follows from considerations in I that a disperse system in the equilibrium state may be visualized simply as a uniform cloud of particles suspended by the upward flow of a fluid. Pseudo-turbulence in such a state is axially symmetric, the axis of symmetry coinciding with the direction of the flow. Using the reference frame whose axis r_1 is directed along the axis of symmetry and separating the variables in (2.2), we get

$$A_{jj}\frac{d^2 f_j^0}{dw_j'^2} - c_{jj}^0 w_j' \frac{df_j^0}{dw_j'} - (c_{jj}^0 - e_j) f_j^0 = 0, \tag{2.3}$$

$$f^0 = \prod_{j=1}^{3} f_j^0, \quad \sum_{j=1}^{3} e_j = 0, \quad e_j = \text{constant} \quad (j = 1, 2, 3).$$

(Summation over j is not performed here; it is obvious that the tensors **A**, $\mathbf{c}^0$ and $\boldsymbol{\theta}^0$ are diagonal in the co-ordinate system used.) Equations (2.3) can be reduced to the Hermite equation, so that their unique solution compatible with the requirement (1.5) has the form

$$f_j^0 = C_j \exp\left(-w_j'^2/2\theta_{jj}^0 + \Delta_j w_j'\right) \quad (j = 1, 2, 3), \tag{2·4}$$

where C_j are constants and the quantities θ_{jj}^0 and Δ_j satisfy the relationships.

$$A_{jj} + \theta_{jj}^0 c_{jj}^0 = 0, \quad A_{jj}\Delta_j^2 - c_{jj}^0 \Delta_j + e_j = 0, \quad (j = 1, 2, 3). \tag{2.5a, b}$$

From the first equation we get the tensor equality

$$\mathbf{A} = -\boldsymbol{\theta}^0 \mathbf{c}^0, \tag{2.6}$$

which can be looked upon as the rigorous definition of the tensor of the diffusion in the velocity space **A**. From (2.5*b*) one obtains an expression for e_j in terms of

Δ_j. Keeping in mind that $\mathbf{w}'$ is a random vector whose mean value is zero by definition, we get $\Delta_j = 0$ from (2.4) and, further, $e_j = 0$ from (2.5).

Normalizing the function f^0 from (2.3) and (2.4) with the number concentration of particles, we have finally

$$f^0 = \frac{n}{(8\pi^3\theta^0_{11}\theta^0_{22}\theta^0_{33})^{\frac{1}{2}}} \exp\left(-\sum_{j=1}^{3}\frac{w_j'^2}{2\theta^0_{jj}}\right). \tag{2.7}$$

Thus, according to the above assumptions and to equations (2.6) and (2.7), the quantities A and f^0 can also be considered as known functions.

3. The formal system of successive approximations

Let us now consider a real state of a disperse system which differs from the equilibrium one so that the dynamic variables describing the mean flow depend upon time and space co-ordinates and, consequently, their derivatives do not equal zero. Nevertheless, we assume, as in the similar situation in the conventional kinetic theory (Chapman & Cowling 1952), that the deviation from the equilibrium state is sufficiently small for the following inequalities to be valid:

$$L\frac{\partial \ln\langle\phi\rangle}{\partial r_j} \ll 1, \quad T\frac{\partial \ln\langle\phi\rangle}{\partial t} \ll 1, \tag{3.1}$$

where L and T are the space and time scales of pseudo-turbulence and $\langle\phi\rangle$ denotes any dynamic variable.

The primary problem consists of the determination of such a solution of (1.3) for a non-equilibrium state which is compatible with (1.1). Bearing in mind (3.1), we seek this solution in the form of a series where the role of a small parameter is played by small ratios of the pseudo-turbulent scales to those for the mean flow. To see the order of magnitude of various terms occurring in the corresponding expansions of (1.1) and (1.3) we multiply each term by the factor ϵ^m, where m is the order of the derivative of a dynamic variable involved in this term. It is relevant to bear in mind that those terms in (1.1) and (1.3) which do not include such derivatives altogether have an order of magnitude ϵ. (One can see this from (2.1), which are valid at the equilibrium state.) It should be remarked that the parameter ϵ has a rather formal meaning and is introduced only for convenience in order to derive the equations of successive approximations in a more understandable and straightforward way. Obviously, it must be put equal to unity at the end of calculation. Exactly the same situation is encountered in kinetic theory when the Chapman–Enskog method is applied to the solution of the kinetic equation for gas molecules (Chapman & Cowling 1952). Therefore there is no need to consider it in more detail.

Further, we confine ourselves to the study of a disperse system in the random-phase approximation (see discussion in I). In this case the quantity f can be regarded as an implicit function of t and $\mathbf{r}$, i.e. it depends upon them only through the dynamic variables. Then

$$\frac{Df}{Dt} = \sum_{\phi}\frac{\partial f}{\partial\langle\phi\rangle}\frac{D\langle\phi\rangle}{Dt}, \quad \frac{\partial f}{\partial\mathbf{r}} = \sum_{\phi}\frac{\partial f}{\partial\langle\phi\rangle}\frac{\partial\langle\phi\rangle}{\partial\mathbf{r}}, \tag{3.2}$$

where summation is carried out over all the dynamic variables $\langle\phi\rangle$.

Representing f in the form of a series

$$f = \Sigma \epsilon^m f_m \quad (0 \leqslant m < \infty) \tag{3.3}$$

and putting into action the corresponding expansions for pseudo-turbulent quantities

$$\boldsymbol{\theta} = \Sigma \epsilon^m \boldsymbol{\theta}^{(m)}, \quad \langle \phi' \psi' \rangle = \Sigma \epsilon^m \langle \phi' \psi' \rangle^{(m)}, \quad \boldsymbol{\theta}^{(m)} = \frac{1}{n} \int (\mathbf{w}' * \mathbf{w}') f_m \, d\mathbf{w}', \tag{3.4}$$

we have from (1.2)

$$\left.\begin{aligned} \langle \phi' \psi' \rangle^{(m)} &= R[\phi' \psi'] \, \theta^{(m)}, \quad \langle \phi' \psi'_j \rangle^{(m)} = R_i[\phi, \boldsymbol{\psi}] \, \theta^{(m)}_{ij}, \\ \langle \phi'_i \psi'_j \rangle^{(m)} &= R_{ik}[\boldsymbol{\phi}, \boldsymbol{\psi}] \, \theta^{(m)}_{kj}, \quad \theta^{(m)} = \operatorname{tr} \boldsymbol{\theta}^{(m)} = \theta^{(m)}_{ii}. \end{aligned}\right\} \tag{3.5}$$

Hence, making use of the definitions of $\mathbf{T}$, $\mathbf{E}$, $\mathbf{q}$, $\mathbf{h}$ in (1.1) and using, as an example, the formulae for $\langle \mathbf{H} \rangle$ and $\mathbf{c}$ from I, we obtain the expansions

$$\left.\begin{aligned}
&\mathbf{T} = \Sigma \epsilon^m \mathbf{T}^{(m)}, \quad \mathbf{E} = \Sigma \epsilon^m \mathbf{E}^{(m)}, \quad \mathbf{q} = \Sigma \epsilon^m \mathbf{q}^{(m)}, \quad \mathbf{h} = \Sigma \epsilon^m \mathbf{h}^{(m)}, \\
&\langle \mathbf{H} \rangle = \Sigma \epsilon^m \langle \mathbf{H} \rangle^{(m)}, \quad \mathbf{c} = \Sigma \epsilon^m \mathbf{c}^{(m)}, \quad \mathbf{q}^{(m)} = -\langle \rho' \mathbf{v}' \rangle^{(m)}, \\
&\mathbf{T}^{(m)} = \frac{1}{1 - \langle \rho \rangle} \left\{ \frac{\partial \mathbf{q}^{(m)}}{\partial t} + \left(\mathbf{q}^{(m)} \frac{\partial}{\partial \mathbf{r}} \right) \langle \mathbf{v} \rangle + \frac{\partial}{\partial \mathbf{r}} (\langle \mathbf{v} \rangle * \mathbf{q}^{(m)}) \right. \\
&\qquad \left. + \frac{\partial}{\partial \mathbf{r}} [(1 - \langle \rho \rangle) \langle \mathbf{v}' * \mathbf{v}' \rangle^{(m)}] \right\}, \\
&\mathbf{E}^{(m)} = \left(S \delta_{0m} + \frac{1}{2} \frac{d^2 S}{d \langle \rho \rangle^2} \langle \rho'^2 \rangle^{(m)} \right) \langle \mathbf{e} \rangle + \frac{dS}{d \langle \rho \rangle} \langle \rho' \mathbf{e}' \rangle^{(m)}, \\
&\langle \mathbf{H} \rangle^{(m)} = \left[\mathbf{g} + \chi (\beta_1 K_1 \langle \mathbf{u} \rangle + \beta_2 K_2 \langle u \rangle \langle \mathbf{u} \rangle) + \frac{D}{Dt} (\eta \langle \mathbf{u} \rangle) - \frac{1}{d_1} \frac{\partial \langle p \rangle}{\partial \mathbf{r}} \right] \delta_{0m} \\
&\qquad + \beta_1 \left(\frac{dK_1}{d \langle \rho \rangle} \langle \rho' \mathbf{u}' \rangle^{(m)} + \frac{1}{2} \frac{d^2 K_1}{d \langle \rho \rangle^2} \langle \rho'^2 \rangle^{(m)} \langle \mathbf{u} \rangle \right) + \beta_2 \left[K_2 (\langle (\mathbf{u}_0 \mathbf{u}') \mathbf{u}' \rangle^{(m)} \right. \\
&\qquad + \tfrac{1}{2} \mathbf{u}_0 \langle u'^2 - (\mathbf{u}_0 \mathbf{u}')^2 \rangle^{(m)}) + \frac{dK_2}{d \langle \rho \rangle} (\langle u \rangle \langle \rho' \mathbf{u}' \rangle^{(m)} + \langle \rho' (\mathbf{u}_0 \mathbf{u}') \rangle^{(m)} \langle \mathbf{u} \rangle) \\
&\qquad \left. + \frac{1}{2} \frac{d^2 K_2}{d \langle \rho \rangle^2} \langle \rho'^2 \rangle^{(m)} \langle u \rangle \langle \mathbf{u} \rangle \right] + \frac{D}{Dt} \left(\frac{d\eta}{d \langle \rho \rangle} \langle \rho' \mathbf{u}' \rangle^{(m)} \right. \\
&\qquad \left. + \frac{1}{2} \frac{d^2 \eta}{d \langle \rho \rangle^2} \langle \rho'^2 \rangle^{(m)} \langle \mathbf{u} \rangle \right) + \left\langle \left(\mathbf{w}' \frac{\partial}{\partial \mathbf{r}} \right) \left(\eta \mathbf{u}' + \frac{d\eta}{d \langle \rho \rangle} \rho' \langle \mathbf{u} \rangle \right) \right\rangle^{(m)}, \\
&\mathbf{u}_0 = \langle \mathbf{u} \rangle / \langle u \rangle, \quad \mathbf{c}^{(m)} = \| c^{(m)}_{ij} \|, \\
&c^{(m)}_{ij} = \chi \{ [\beta_1 K_1 + \beta_2 K_2 \langle u \rangle] \delta_{ij} + \beta_2 K_2 \langle u \rangle \delta_{i1} \delta_{j1} \} \delta_{0m} - \chi \frac{\partial}{\partial r_j} (\eta \langle u_i \rangle) \delta_{m1}, \\
&\mathbf{h}^{(m)} = \left(1 - \frac{1}{\chi} \frac{\langle \rho \rangle}{(1 - \langle \rho \rangle)} \right) \mathbf{g} \, \delta_{0m} - \frac{1}{\chi} \frac{1}{(1 - \langle \rho \rangle)} \langle \mathbf{H} \rangle^{(m)}.
\end{aligned}\right\} \tag{3.6}$$

Here the notation of I is used as pointed out above.

In order to develop a self-consistent procedure of successive approximations it is necessary to define functions $f^{(m)}$ in such a manner as to yield just the Mth approximation for the distribution function and for all the dynamic variables, neglecting all the terms of the expansions (3.3) and (3.4) whose numbers are greater than M. The meaning of this requirement is the same as that in the kinetic theory of gases (Chapman & Cowling 1952). To this end, we represent the convective derivatives of f and of any dynamic variable $\langle \phi \rangle$ in the following form:

$$\frac{Df}{Dt} = \Sigma \epsilon^m \frac{D_m f}{Dt}, \quad \frac{D \langle \phi \rangle}{Dt} = \Sigma \epsilon^m \frac{D_m \langle \phi \rangle}{Dt}. \tag{3.7}$$

One needs to determine operators in these expansions so that the expansions agree with the laws of mass and momentum conservation of both phases, i.e. with (1.1). This can be achieved if one takes the various operators in (3.7) in the form

$$\left.\begin{aligned}\frac{D_m\langle\rho\rangle}{Dt} &= -\left[\langle\mathbf{u}\rangle\frac{\partial\langle\rho\rangle}{\partial\mathbf{r}}+(1-\langle\rho\rangle)\frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}}\right]\delta_{0m}+\frac{\partial\mathbf{q}^{(m)}}{\partial\mathbf{r}},\\ \frac{D_m\langle\mathbf{w}\rangle}{Dt} &= -\frac{1}{\langle\rho\rangle}\frac{\partial(\langle\rho\rangle\boldsymbol{\theta}^{(m)})}{\partial\mathbf{r}}+\langle\mathbf{H}\rangle^{(m)},\\ \frac{D_m\langle\mathbf{v}\rangle}{Dt} &= -\left[\left(\langle\mathbf{u}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{v}\rangle+\frac{1}{d_0(1-\langle\rho\rangle)}\frac{\partial\langle p\rangle}{\partial\mathbf{r}}\right]\delta_{0m}-\mathbf{T}^{(m)}+\frac{2\nu_0}{1-\langle\rho\rangle}\frac{\partial\mathbf{E}^{(m-1)}}{\partial\mathbf{r}}+\mathbf{h}^{(m)}.\end{aligned}\right\} \tag{3.8}$$

Besides, from the first equation of (1.1) we have the following:

$$\frac{\partial\langle\mathbf{w}\rangle}{\partial\mathbf{r}} = \Sigma\epsilon^m\,\mathrm{div}_m\langle\mathbf{w}\rangle,\quad \mathrm{div}_m\langle\mathbf{w}\rangle = -\frac{D_m\langle\rho\rangle}{Dt}. \tag{3.9}$$

Equation (3.9) is similar to (3.7) and (3.8), and completes them in a sense.

It is permissible to define the convective derivative of the mean pressure in the form

$$\frac{D\langle p\rangle}{Dt} = \frac{D_0\langle p\rangle}{Dt},\quad \frac{D_m\langle p\rangle}{Dt} = 0,\quad (m>0). \tag{3.10}$$

Equations (3.3) and (3.7) make it possible to express the convective derivative of f as follows:

$$\frac{Df}{Dt} = \sum_{\phi}\left(\Sigma\epsilon^m\frac{D_m\langle\phi\rangle}{Dt}\right)\left(\Sigma\epsilon^m\frac{\partial f_n}{\partial\langle\phi\rangle}\right). \tag{3.11}$$

By making use of (3.8)–(3.10), it is not difficult to obtain the explicit representations for terms of the first sum in (3.7). For example,

$$\left.\begin{aligned}\frac{D_0 f}{Dt} &= \sum_{\phi}\frac{\partial f_0}{\partial\langle\phi\rangle}\frac{D_0\langle\phi\rangle}{Dt},\\ \frac{D_1 f}{Dt} &= \sum_{\phi}\left(\frac{Df_0}{\partial\langle\phi\rangle}\frac{D_1\langle\phi\rangle}{Dt}+\frac{Df_1}{\partial\langle\phi\rangle}\frac{D_0\langle\phi\rangle}{Dt}\right),\quad \text{etc.}\end{aligned}\right\} \tag{3.12}$$

These relationships are similar to those in the kinetic theory of gases. It is convenient to choose the equilibrium distribution function f^0 from (2.7) as the zeroth approximation f_0 in (3.3). Then we have from (1.3) the following equation for subsequent terms in (3.3):

$$\begin{aligned}\sum_{j=1}^{3}A_{jj}\left[\frac{\partial^2 f_m}{\partial w_j'^2}+\frac{1}{\theta_{jj}^{(0)}}\frac{\partial(w_j' f_m)}{\partial w_j'}\right] &= -\frac{D_{m-1}f}{Dt}-\mathbf{w}'\frac{\partial f_{m-1}}{\partial\mathbf{r}}\\ &+\frac{\partial}{\partial\mathbf{w}'}(\mathbf{c}^{(1)}\mathbf{w}'f_{m-1})-\frac{1}{\langle\rho\rangle}\sum_{n=0}^{m-1}\frac{\partial(\langle\rho\rangle\boldsymbol{\theta}^{(n)})}{\partial\mathbf{r}}\frac{\partial f_{m-n-1}}{\partial\mathbf{w}'}\\ &+\left(\frac{\partial f_{m-1}}{\partial\mathbf{r}}*\mathbf{w}'\right):\left(\frac{\partial}{\partial\mathbf{r}}*\langle\mathbf{w}\rangle\right).\end{aligned} \tag{3.13}$$

Here we use (1.2) and (2.6), take into account the equalities $\mathbf{c}^{(m)} = 0$ $(m > 1)$ following from (3.6) and use the reference frame whose axes coincide with the principal axes of the tensor **A**.

The solution of (3.13) for any m must meet the requirement

$$\int f_m d\mathbf{w}' = \int w'_j f_m d\mathbf{w}' = 0, \tag{3.14}$$

which is the necessary condition of consistency of these solutions with the conservation equations (1.1) written in the same approximation. We have in the mth approximation

$$\left.\begin{aligned} &\frac{D\langle\rho\rangle}{Dt} + \langle\rho\rangle\frac{\partial\langle\mathbf{w}\rangle}{\partial\mathbf{r}} = 0, \\ d_1\langle\rho\rangle\frac{D\langle\mathbf{w}\rangle}{Dt} &= -d_1\frac{\partial}{\partial\mathbf{r}}\left(\langle\rho\rangle\sum_{n=0}^{m}\boldsymbol{\theta}^{(n)}\right) + d_1\langle\rho\rangle\sum_{n=0}^{m}\langle\mathbf{H}\rangle^{(n)}, \\ &\frac{D_v\langle\rho\rangle}{Dt} - (1-\langle\rho\rangle)\frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}} - \frac{\partial}{\partial\mathbf{r}}\sum_{n=0}^{m}\mathbf{q}^{(n)} = 0, \\ d_0(1-\langle\rho\rangle)\frac{D_v\langle\mathbf{v}\rangle}{Dt} &= -\frac{\partial\langle p\rangle}{\partial\mathbf{r}} - d_0(1-\langle\rho\rangle)\sum_{n=0}^{m}\mathbf{T}^{(n)} + 2\mu_0\sum_{n=0}^{m-1}\frac{\partial}{\partial\mathbf{r}}\mathbf{E}^{(n)} \\ &\quad + d_0(1-\langle\rho\rangle)\sum_{n=0}^{m}\mathbf{h}^{(n)}, \qquad \frac{D_v}{Dt} = \frac{\partial}{\partial t} + \langle\mathbf{v}\rangle\frac{\partial}{\partial\mathbf{r}}. \end{aligned}\right\} \tag{3.15}$$

Various pseudo-turbulent quantities appearing in these equations are functions of the dynamic variables and can be expressed in an explicit form by means of (3.4)–(3.6), equations (3.13) for all f_n $(n < m)$ being solved.

To conclude this section, we discuss briefly similarities and distinctions between the above method and the known Chapman–Enskog method. It is clear that the procedure of the successive approximations is the same in either case, however, one essential difference must be noted. In the latter method, the scalar $\theta = \operatorname{tr}\boldsymbol{\theta}$ characterizes the gas temperature and is regarded as an independent parameter. On the other hand, the components of the tensor $\boldsymbol{\theta}$ in the flow of a disperse system are determined entirely by the properties of the mean flow. They are functions of the dynamic variables and cannot be considered as independent quantities. In this respect they differ from the mean concentration or the mean velocity of the dispersed phase. Therefore, the independent consideration of the transfer equation for $\boldsymbol{\theta}$ while solving the kinetic equation (1.3) seems to be unnecessary in the case under study.

4. The Eulerian and Navier–Stokes approximations

Solving (3.13) at various m in turn, we can in principle derive the dynamic equations (3.15) to any accuracy. It seems to be quite sufficient in practice to find only equations of the zeroth and the first approximations to which it is relevant to refer, by analogy with kinetic theory and one-phase hydrodynamics, to as Eulerian and Navier–Stokes approximations, respectively.

The conservation equations of the Eulerian approximation can be derived very simply. In fact, they are obtained from (3.15), the distribution function being given by (2.7) and all the pseudo-turbulent equantities involved being represented in terms of the dynamic variables in the same manner as for the equilibrium state. Keeping in mind the axial symmetry of equilibrium pseudo-turbulence, we can conclude that $\mathbf{q}^{(0)} = \{q^{(0)}, 0, 0\}$ and the tensors $\boldsymbol{\theta}^{(0)}$ and $\mathbf{T}^{(0)}$ are diagonal. Therefore, only normal pseudo-turbulent stresses of both phases and only the pseudo-turbulent flux of the fluid in the direction of the mean relative velocity $\langle\mathbf{u}\rangle$ are taken into account in this approximation. The nature of those normal stresses is similar, in a sense, to that of the pressure in a gas. The additional flux $\mathbf{q}^{(0)}$ is analogous to the flux which arises in a turbulent flow of a one-phase compressible fluid.

Let us now consider the next approximation and introduce a new unknown function g_1 with

$$f_1 = f_0 g_1, \quad f_0 = f^0, \quad \langle\phi'\psi'\rangle^{(0)} = \langle\phi'\psi'\rangle^0. \tag{4.1}$$

Equation (3.13) at $m = 1$ can be rewritten as follows:

$$\left.\begin{aligned}
&\sum_{j=1}^{3} A_{ij}\left(\frac{\partial^2}{\partial w_j'^2} - \frac{w_j'}{\theta_{jj}^0}\frac{\partial}{\partial w_j'}\right) g_1 = -\frac{1}{f^0}\left[\frac{D_0 f}{Dt} + \mathbf{w}'\frac{\partial f^0}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{w}'}(\mathbf{c}^{(1)}\mathbf{w}' f^0)\right.\\
&\quad\left. + \frac{1}{\langle\rho\rangle}\frac{\partial(\langle\rho\rangle\boldsymbol{\theta}^0)}{\partial \mathbf{r}}\frac{\partial f^0}{\partial \mathbf{w}'} - \left(\frac{\partial f^0}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \left(\frac{\partial}{\partial \mathbf{r}} * \langle\mathbf{w}\rangle - \mathbf{C}\right) - \left(\frac{\partial f^0}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \mathbf{C}\right],\\
&\mathbf{C} = \|C_{ij}\|, \quad C_{ij} = \frac{\partial\langle w_i\rangle}{\partial r_i}\delta_{ij}, \quad \sum_{i=1}^{3}\frac{\partial\langle w_i\rangle}{\partial r_i} = \operatorname{div}_0\langle\mathbf{w}\rangle,
\end{aligned}\right\} \tag{4.2}$$

where $\operatorname{div}_0\langle\mathbf{w}\rangle$ is determined by (3.9).

Transforming terms in the right-hand side of (4.2) with the help of relationships of the preceding sections, we obtain the equations

$$\left.\begin{aligned}
&\frac{1}{f^0}\frac{D_0 f}{Dt} = -\operatorname{div}_0\langle\mathbf{w}\rangle + \frac{1}{2}\sum_{j=1}^{3}\left(\frac{w_j'^2}{\theta_{jj}^0} - 1\right)\frac{D_0 \ln\theta_{jj}^0}{Dt},\\
&\frac{\mathbf{w}'}{f^0}\frac{\partial f^0}{\partial \mathbf{r}} = \mathbf{w}'\frac{\partial \ln\langle\rho\rangle}{\partial \mathbf{r}} + \frac{1}{2}\sum_{j=1}^{3}\left(\frac{w_j'^2}{\theta_{jj}^0} - 1\right)\mathbf{w}'\frac{\partial \ln\theta_{jj}^0}{\partial \mathbf{r}},\\
&\frac{1}{\langle\rho\rangle f^0}\frac{\partial(\langle\rho\rangle\boldsymbol{\theta}^0)}{\partial \mathbf{r}}\frac{\partial f^0}{\partial \mathbf{w}'} = -\mathbf{w}'\frac{\partial \ln\langle\rho\rangle}{\partial \mathbf{r}} - \sum_{j=1}^{3} w_j'\frac{\partial \ln\theta_{jj}^0}{\partial r_j},\\
&-\frac{1}{f^0}\frac{\partial}{\partial \mathbf{w}'}(\mathbf{c}^{(1)}\mathbf{w}' f^0) = \sum_{j=1}^{3} c_{jj}^{(1)}\left(\frac{w_j'^2}{\theta_{jj}^0} - 1\right) + \mathbf{w}'\boldsymbol{\Gamma}^{(1)}\mathbf{w}',\\
&-\frac{1}{f^0}\left(\frac{\partial f^0}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \left(\frac{\partial}{\partial \mathbf{r}} * \langle\mathbf{w}\rangle - \mathbf{C}\right) = \mathbf{w}'\boldsymbol{\Gamma}^{(2)}\mathbf{w}',\\
&-\frac{1}{f^0}\left(\frac{\partial f^0}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \mathbf{C} = \sum_{j=1}^{3}\frac{w_j'^2}{\theta_{jj}^0}\frac{\partial\langle w_j\rangle}{\partial r_j}, \quad \boldsymbol{\Gamma}^{(k)} = \|\Gamma_{ij}^{(k)}\|,\\
&\Gamma_{ij}^{(1)} = \frac{1}{2}\left(\frac{c_{ij}^{(1)}}{\theta_{ii}^0} + \frac{c_{ji}^{(1)}}{\theta_{jj}^0}\right) - \frac{c_{ii}^{(1)}}{\theta_{ii}^0}\delta_{ij},\\
&\Gamma_{ij}^{(2)} = \frac{1}{2}\left(\frac{1}{\theta_{ii}^0}\frac{\partial\langle w_i\rangle}{\partial r_j} + \frac{1}{\theta_{jj}^0}\frac{\partial\langle w_j\rangle}{\partial r_i}\right) - \frac{1}{\theta_{ii}^0}\frac{\partial\langle w_i\rangle}{\partial r_i}\delta_{ij}.
\end{aligned}\right\} \tag{4.3}$$

Introducing these relations into (4.2), we get

$$\left.\begin{aligned}
&\sum_{j=1}^{3} A_{jj}\left(\frac{\partial^2}{\partial w_j'^2}-\frac{w_j'}{\theta_{jj}^0}\frac{\partial}{\partial w_j'}\right) g_1 = \sum_{j=1}^{3}\left(1-\frac{w_j'^2}{\theta_{jj}^0}\right)\left(\frac{1}{2}\frac{D_0 \ln\theta_{jj}^0}{Dt}+c_{jj}^{(1)}\right.\\
&\quad \left.+\frac{\partial\langle w_j\rangle}{\partial r_j}\right) - \mathbf{w}'\boldsymbol{\Gamma}\mathbf{w}' + \frac{1}{2}\sum_{j=1}^{3}\left(1-\frac{w_j'^2}{\theta_{jj}^0}\right)\mathbf{w}'\frac{\partial \ln\theta_{jj}^0}{\partial \mathbf{r}} + \sum_{j=1}^{3} w_j'\frac{\partial\ln\theta_{jj}^0}{\partial r_j},\\
&\qquad \boldsymbol{\Gamma} = \|\Gamma_{ij}\|, \quad \Gamma_{ij} = \Gamma_{ij}^{(1)}+\Gamma_{ij}^{(2)},\\
&\Gamma_{ij} = \frac{1}{2}\left(\frac{1}{\theta_{ii}^0}\frac{\partial W_i}{\partial r_j}+\frac{1}{\theta_{jj}^0}\frac{\partial W_j}{\partial r_i}\right)-\frac{1}{\theta_{jj}^0}\frac{\partial W_i}{\partial r_i}\delta_{ij}, \quad \mathbf{W} = \langle\mathbf{w}\rangle+\chi\eta\langle\mathbf{u}\rangle.
\end{aligned}\right\} \tag{4.4}$$

Further, we seek a partial solution of (4.4) in the form

$$g_1 = K + \sum_{j=1}^{3} L_j w_j' + \sum_{i,j=1}^{3} (M_{ij}+N_{ij}w_j')\, w_i' w_j', \tag{4.5}$$

where K, L_j, M_{ij} and N_{ij} do not depend upon $\mathbf{w}'$. Substituting (4.5) into (4.4) and equating terms of the corresponding order with respect to w_i' and w_j' in the left-hand and the right-hand sides of the equation obtained, we get the following relationships:

$$\begin{aligned}
N_{ij} &= -\frac{1}{2\theta_{jj}^0\,(c_{ii}^{(0)}+2c_{jj}^{(0)})}\frac{\partial\ln\theta_{jj}^0}{\partial r_i},\\
M_{ij} &= -\frac{\Gamma_{ij}}{c_{ii}^{(0)}+c_{jj}^{(0)}} \quad (i \neq j),\\
M_{jj} &= \frac{1}{2A_{jj}}\left(\frac{1}{2}\frac{D_0\ln\theta_{jj}^0}{Dt}+\frac{\partial\langle w_j\rangle}{\partial r_j}+c_{jj}^{(1)}\right)\\
L_j &= \frac{1}{2}\frac{1}{c_{jj}^{(0)}}\frac{\partial\ln\theta_{jj}^0}{\partial r_j}+\frac{1}{2}\sum_{i=1,\,i\neq j}^{3}\frac{1}{2c_{ii}^{(0)}+c_{jj}^{(0)}}\frac{\partial\ln\theta_{ii}^0}{\partial r_j}.
\end{aligned} \tag{4.6}$$

Making use of the first condition in (3.14), we obtain after simple calculation the expression for the quantity K in (4.5):

$$K = \frac{1}{2}\sum_{j=1}^{3}\left(\frac{1}{2}\frac{D_0\ln\theta_{jj}^0}{Dt}+\frac{\partial\langle w_j\rangle}{\partial r_j}+c_{jj}^{(1)}\right)\frac{1}{c_{jj}^{(0)}}. \tag{4.7}$$

The derivatives of $\ln\theta_{jj}^0$ involved in the above formulae can be expressed in the obvious form

$$\left.\begin{aligned}
\frac{D_0\ln\theta_{jj}^0}{Dt} &= \frac{\partial\ln\theta_{jj}^0}{\partial\langle\rho\rangle}\frac{D_0\langle\rho\rangle}{Dt}+\frac{\partial\ln\theta_{jj}^0}{\partial\langle u\rangle}\frac{D_0\langle u\rangle}{Dt},\\
\frac{\partial\ln\theta_{jj}^0}{\partial\mathbf{r}} &= \frac{\partial\ln\theta_{jj}^0}{\partial\langle\rho\rangle}\frac{\partial\langle\rho\rangle}{\partial\mathbf{r}}+\frac{\partial\ln\theta_{jj}^0}{\partial\langle u\rangle}\frac{\partial\langle u\rangle}{\partial\mathbf{r}},
\end{aligned}\right\} \tag{4.8}$$

where the representations of the convective derivatives of $\langle\rho\rangle$ and $\langle u\rangle$ follow from (3.8). Note that the quantites θ_{jj}^0 are known functions depending only upon two dynamic variables, $\langle\rho\rangle$ and $\langle u\rangle$, as can be readily shown from the analysis of equilibrium pseudo-turbulence in I. One can check also that the second requirement of (3.14) is satisfied identically.

The components of the tensor $\boldsymbol{\theta}^{(1)}$ are expressed in the form

$$\theta_{ij}^{(1)} = \frac{1}{n}\int w_i' w_j' f_1 dw'$$
$$= \sum_{k=1}^{3} \frac{1+\delta_{ik}}{c_{kk}^{(0)}} \left(\frac{1}{2}\frac{D_0 \ln \theta_{kk}^0}{Dt} + \frac{\partial \langle w_k \rangle}{\partial r_k} + c_{kk}^{(1)}\right) \theta_{ii}^0 \delta_{ij} + \theta_{ii}^0 \theta_{jj}^0 \left(\frac{\Gamma_{ij}}{c_{ii}^{(0)} + c_{jj}^{(0)}}\right). \quad (4.9)$$

Equation (4.9) together with (3.5) and (3.6) defines completely the conservation equations for the mean flow in the Navier–Stokes approximation, which results from (3.15) at $m = 1$. New terms occur in these equations in addition to those in the Eulerian equations. These terms are proportional to the derivatives of the dynamic variables. First of all, the components of $\mathbf{q}$ which are perpendicular to the vector $\langle \mathbf{u} \rangle$ arise. Also, the normal stresses in both phases undergo some change and (which is especially important) additional tangential pseudo-turbulent stresses appear in the equations discussed. Their origin is similar to that of the viscous stresses in a gas or of the Reynolds stresses in a turbulent flow. Finally, the forces $\langle \mathbf{H} \rangle$ and $\mathbf{h}$ involved in these equations are also changed; new terms proportional to derivatives of the dynamic variables appear in the expressions for them.

Representations of the tangential stresses differ essentially from those postulated by the various phenomenological models of disperse systems. In particular, in the general case it is not an easy task to single out the tensors of effective viscosity, and the mean rates of deformation from the tensors of tangential stress. As can be readily seen, it is possible only for the simplest flows of a disperse system (e.g. for one-dimensional flow). The effective pseudo-turbulent viscosities of both phases depend substantially upon orientation and the general type of a flow, so that it is no use to consider them as the permanent physical properties of the given disperse system.†

Equations (3.5) for $m = 0$ or $m = 1$ are far more complex than the common Eulerian or Navier–Stokes equations for one-phase medium. In the first place, this is due to the fact that these equations contain not only the usual nonlinear inertial terms, but also other strong nonlinearities resulting from intricate dependence of pseudo-turbulent stresses and other pseudo-turbulent quantities involved in them upon the dynamic variables. Obviously, these nonlinearities are especially significant when the pseudo-turbulent motion is developed sufficiently. Simple estimations show that this is the case even for the disperse systems whose concentration exceeds 0·01–0·05, except for suspensions of very small particles in a highly viscous fluid and for the suspensions characterized by nearly equal densities of the particles and the fluid. However, even then the allowance for pseudo-turbulence is important from the main point of view because in the opposite case one should have to postulate in some cases an additional 'constitutive' equation or an assumption for determination of the concentration. We can refer, for example, to a steady one-dimensional flow in a vertical tube. If pseudo-turbu-

† It should be noted that this remark is true only for the viscosities resulting from the pseudo-turbulent motion. There also exists the other viscosity associated with a regular fluid flow through a lattice of unmovable particles without pulsation (e.g. the Einstein viscosity), which ought not, of course, to depend upon the properties of a flow.

lence is not taken into account, then there are only two equations for $\langle v \rangle$ and $\langle w \rangle$ which are not satisfied identically, so that the distribution of $\langle \rho \rangle$ over the tube cross-section is unspecified and the solution exists for any function $\langle \rho \rangle$. On the other hand equations (3.15) without pseudo-turbulent terms have no solution for a continuous distribution of $\langle \rho \rangle$, if such a flow occurs in an oblique tube. It can be shown that these difficulties are eliminated if one takes account of the pseudo-turbulent motion of the particles and the fluid.

It would be appropriate to discuss in more detail the equations obtained and the formulation of boundary conditions for them in close connexion with the solution of various concrete problems of two-phase flow, etc. Therefore we do not consider these questions here.

The treatment of this paper makes the theory proposed in I more complete and contributes some new matter to it. Up to this point, we have considered no particular problems or numerical examples which could illustrate applications of the theory. Therefore, one can consider this theory at present being rather formal. However, from now on we have a sufficient information to proceed to concrete problems immediately. For present examples, we hope to investigate in the next part of the paper the diffusivities and the viscosities of both phases as well as other pseudo-turbulent characteristics and rheological properties of suspensions at small Reynolds number.

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